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1. Iterative Substructuring with Lagrange Multipliers for Coupled Fluid-Solid Scattering

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1. Introduction. In [9], we have proposed an iterative method for the solution of linear systems arizing from finite element discretization of the time harmonic acoustics of coupled fluid-solid systems in fluid pressure and solid displacement formulation. The method extended the FETI-H method for the Helmholtz equation [4, 6, 7, 12] to coupled fluid-elastic acoustics. In this paper, we investigate a stabilization of the discrete coupled system for the case when the solid scatterer is at resonance and investigate computationally the convergence of the iterative substructuring method for the modified system.

The main idea of the method of [9] is as follows. The fluid and the solid domains are decomposed into non-overlapping subdomains. Continuity of the solution is enforced by Lagrange multipliers. To prevent singular or nearly singular subdomain matrices due to resonance, the continuity conditions betweeen the subdomains are replaced by artificial radiation-like conditions. Because original degrees of freedom are coupled across the wet interface, the system is augmented by duplicating the degrees of freedom on the wet interface and adding equations enforcing the equality of the original and the duplicate degrees of freedoom. The original degrees of freedom can then be eliminated subdomain by subdomain and the resulting system is solved by Krylov iterations preconditioned by a Galerkin correction on a subspace consisting of plane waves in each subdomain. In each iteration, the method requires the solution of one independent acoustic problem per subdomain, and the solution of a coarse problem with several degrees of freedom per subdomain. The number of iterations in was most cases about the same as the number of iterations of the FETI-H method for the related Helmholtz problem with Neumann boundary condition instead of an elastic scatterer, which was explained by numerical decoupling of the fluid and the elastic fields in the stiff scatterer limit.

In this article, we propose a new artificial radiation-like condition on the wet interface, and we observe in computational tests that that it it prevents deterioration of convergence in the case of one solid subdomain at resonance. We also investigate the sensitivity of the method to variants of artificial radiation condition between the elastic subdomains.

Our radiation-like condition between elastic subdomains has been inspired by [2], which generalized the alternating method of [5] to elasticity. Iterative methods consisting of alternating solution in the fluid and the solid region are known [1, 3]. In [3], the alternating method of [5] was extended to the coupled problem, with the wet interface conditions replaced by their complex linear combinations. The resulting iterative algorithm needs either access to normal derivatives or additional variables on the wet interface. Our radiation-like condition on the wet interface is obtained by a simple modification of the coupled system matrix, resulting in an equivalent algebraic system. Since this process is unrelated to the substructuring method at hand, it may be of independent interest.

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2. The scattering problem. We need to describe the scattering problem and the discretization used. This material is standard [11, 13] and it is included only for completeness and to introduce the notation.

We consider an acoustic scattering problem with an elastic scatterer completely immersed in a fluid. Let Ω and Ω_e be bounded domains in $\Re^n$, $= 2, 3$, $\overline{\Omega}_e \subset \Omega$, and let $\Omega_f = \Omega \setminus \Omega_e$, cf., Figure 5.1. Let ν denote the exterior normal of Ω_e. Let $\partial\Omega$ be decomposed into disjoint subsets, $\partial\Omega = \Gamma_d \cup \Gamma_n \cup \Gamma_a$. The domain Ω_f is filled with a fluid. The acoustic pressure at time t is assumed to be of the form $\mathrm{Re}\, p e^{i\omega t}$, where p is complex amplitude independent of t. The amplitude p is governed by the Helmholtz equation

$$\Delta p + k^2 p = 0 \quad \text{in} \quad \Omega_f, \tag{2.1}$$

with the boundary conditions

$$p = p_0 \text{ on } \Gamma_d, \quad \frac{\partial p}{\partial \nu} = 0 \text{ on } \Gamma_n, \quad \frac{\partial p}{\partial \nu} + ikp = 0 \text{ on } \Gamma_a, \tag{2.2}$$

where $k = \omega/c_f$ is the wave number and c_f is the speed of sound in the fluid. The boundary conditions (2.2) model excitation, sound hard boundary, and outgoing boundary, respectively. The amplitude of the displacement u of the elastic body occupying the domain Ω_e satisfies the elastodynamic equation

$$\nabla \cdot \tau + \omega^2 \rho_e u = 0 \quad \text{in} \quad \Omega_e, \tag{2.3}$$

where τ is the stress tensor and ρ_e is the density of the solid. For simplicity, we consider an isotropic homogeneous material with

$$\tau = \lambda I(\nabla \cdot u) + 2\mu e(u), \quad e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.4}$$

where λ and μ are the Lamé coefficients of the solid.

Let $\Gamma = \partial\Omega_e$ be the wet interface. On Γ, the fluid pressure and the solid displacement satisfy

$$\nu \cdot u = \frac{1}{\rho_f \omega^2} \frac{\partial p}{\partial \nu}, \quad \nu \cdot \tau \cdot \nu = -p, \quad \nu \times \tau \cdot \nu = 0, \tag{2.5}$$

where ρ_f is the fluid density. The interface conditions (2.5) model the continuity of normal displacement, the balance of normal forces, and zero tangential tension, respectively.

We use the following variational form. Define the spaces $V_f = \{q \in H^1(\Omega_f) \mid q = 0 \text{ on } \Gamma_d\}$, $V_e = (H^1(\Omega_e))^n$, where H^1 is the Sobolev space of generalized functions with square integrable generalized first derivatives. Assuming that p_0 on Γ_d is extended to a function in $H^1(\Omega_f)$, multiplying equation (2.1) by a test function $q \in V_f$, equation (2.3) by a test function $u \in V_e$, and integrating by parts, we obtain the following variational form of (2.1) – (2.5): Find p such that $p - p_0 \in V_f$, and $u \in V_e$ such that for all $q \in V_f$ and all $v \in V_e$,

$$-\int_{\Omega_f} \nabla p \nabla q + k^2 \int_{\Omega_f} pq - ik \int_{\Gamma_a} pq - \omega^2 \int_{\Gamma} \rho_f (\nu \cdot u) q = 0,$$

$$-\int_{\Omega_e} \left(\lambda(\nabla \cdot u)(\nabla \cdot v) + 2\mu e(u) : e(v)\right) + \omega^2 \int_{\Omega_e} \rho_e u \cdot v - \int_{\Gamma} p(\nu \cdot v) = 0.$$

We replace V_f and V_e with conforming finite element spaces and obtain the algebraic system

$$\begin{bmatrix} -\mathbf{K}_f + k^2\mathbf{M}_f - ik\mathbf{G}_f & -\rho_f\omega^2\mathbf{T} \\ -\mathbf{T}' & -\mathbf{K}_e + \omega^2\mathbf{M}_e \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{r} \\ 0 \end{bmatrix}. \tag{2.6}$$

In the coupled system (2.6), $\mathbf{p}$ and $\mathbf{u}$ are the vectors of the (values of) degrees of freedom of p and u, i.e., p and u are the finite element interpolations of $\mathbf{p}$ and $\mathbf{u}$, respectively. The matrix blocks in (2.6) are defined by

$$\mathbf{p}'\mathbf{K}_f\mathbf{q} = \int_{\Omega_f} \nabla p \cdot \nabla q, \qquad \mathbf{p}'\mathbf{M}_f\mathbf{q} = \int_{\Omega_f} pq,$$

$$\mathbf{p}'\mathbf{G}_f\mathbf{q} = \int_{\Gamma_a} pq, \qquad \mathbf{u}'\mathbf{K}_e\mathbf{v} = \int_{\Omega_e} \big(\lambda(\nabla\cdot u)(\nabla\cdot v) + 2\mu e(u) : e(v)\big),$$

$$\mathbf{u}'\mathbf{M}_e\mathbf{v} = \int_{\Omega_e} \rho_e(u\cdot v), \qquad \mathbf{p}'\mathbf{T}\mathbf{v} = \int_{\Gamma} p(\nu\cdot v).$$

3. Iterative Substructuring. In this section, we summarize the iterative method following [9]. Further details and a development of the method starting from FETI-H can be found in [9]. The present method differs in the more general choice of artificial radiation condition between elastic subdomains.

The fluid and solid domains are decomposed into nonoverlapping subdomains that consist of unions of elements,

$$\overline{\Omega}_f = \bigcup_{s=1}^{N_f} \overline{\Omega}_e^s, \qquad \overline{\Omega}_e = \bigcup_{s=1}^{N_e} \overline{\Omega}_e^s. \tag{3.1}$$

The fields and vectors of degrees of freedom corresponding to Ω_f^s and Ω_e^s are denoted by p^s, u^s, $\mathbf{p}^s$ and $\mathbf{u}^s$, respectively. The normal vector to $\partial\Omega^s$ is denoted by ν^s.

The Helmholtz equation (2.1) is then equivalent to the same equation in each of the subdomains Ω_f^s, with the interface conditions

$$p^s = p^t, \quad \frac{\partial p^s}{\partial \nu^s} + \frac{\partial p^t}{\partial \nu^t} = 0, \quad \text{on } \partial\Omega_f^s \cap \partial\Omega_f^t. \tag{3.2}$$

Similarly, the elastodynamic equation (2.3) is equivalent to the same equation in each of the subdomains Ω_e^s, with the continuity of the displacement and the traction on the intersubdomain interfaces,

$$u^s = u^t, \quad \tau(u^s)\nu^s + \tau(u^t)\nu^t = 0, \quad \text{on } \partial\Omega_e^s \cap \partial\Omega_e^t. \tag{3.3}$$

The continuity of the pressure and the displacement will be enforced by Lagrange multipliers.

Define subdomain matrices by subassembly,

$$\mathbf{p}^{s\prime}\mathbf{K}_f^s\mathbf{q} = \int_{\Omega_f^s} \nabla p \cdot \nabla q, \qquad \mathbf{p}^{s\prime}\mathbf{M}_f^s\mathbf{q}^s = \int_{\Omega_f^s} pq,$$

$$\mathbf{p}^{s\prime}\mathbf{G}_f^s\mathbf{q}^s = \int_{\partial\Omega_f^s\cap\Gamma_a} pq, \qquad \mathbf{u}^{s\prime}\mathbf{K}_e^s\mathbf{v}^s = \int_{\Omega_e^s} \lambda(\nabla\cdot u)(\nabla\cdot v) + 2\mu e(u) : e(v),$$

$$\mathbf{u}^{s\prime}\mathbf{M}_e^s\mathbf{v} = \int_{\Omega_e^s} \rho_e(u\cdot v), \qquad \mathbf{p}^{r\prime}\mathbf{T}^{rs}\mathbf{v}^s = \int_{\partial\Omega_f^r\cap\partial\Omega_e^s} p(\nu\cdot v).$$

We will use vectors consisting of all subdomain degrees of freedom,

$$\hat{\mathbf{p}} = \begin{bmatrix} \mathbf{p}^1 \\ \vdots \\ \mathbf{p}^{N_f} \end{bmatrix}, \qquad \hat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}^1 \\ \vdots \\ \mathbf{u}^{N_e} \end{bmatrix},$$

and the corresponding partitioned matrices,

$$\hat{\mathbf{K}}_f = \mathrm{diag}(\mathbf{K}_f^s) = \begin{bmatrix} \mathbf{K}_f^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{K}_f^{N_f} \end{bmatrix}, \qquad \hat{\mathbf{K}}_e = \mathrm{diag}(\mathbf{K}_e^s) = \begin{bmatrix} \mathbf{K}_e^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{K}_e^{N_e} \end{bmatrix}.$$

The matrices $\hat{\mathbf{M}}_f$, $\hat{\mathbf{G}}_f$, and $\hat{\mathbf{M}}_e$ are defined similarly, and

$$\hat{\mathbf{T}} = (\mathbf{T}^{rs})_{rs} = \begin{bmatrix} \mathbf{T}^{11} & \dots & \mathbf{T}^{1,N_e} \\ \vdots & \ddots & \vdots \\ \mathbf{T}^{N_f,1} & \dots & \mathbf{T}^{N_f,N_e} \end{bmatrix}.$$

Let $\mathbf{N}_f$ and $\mathbf{N}_e$ be the matrices with $0, 1$ entries of the global to local maps corresponding to the decompositions of Ω_f and Ω_e, respectively, cf., (3.1), so that

$$\mathbf{K}_f = \mathbf{N}_f'\hat{\mathbf{K}}_f\mathbf{N}_f, \qquad \mathbf{K}_e = \mathbf{N}_e'\hat{\mathbf{K}}_e\mathbf{N}_e.$$

Let $\mathbf{B}_f = (\mathbf{B}_f^1, \dots, \mathbf{B}_f^{N_f})$ and $\mathbf{B}_e = (\mathbf{B}_e^1, \dots, \mathbf{B}_e^{N_e})$ be matrices of full rank such that the conditions $\mathbf{B}_f\hat{\mathbf{p}} = 0$ and $\mathbf{B}_e\hat{\mathbf{u}} = 0$ express the constraint that the values of the same degrees of freedom on two different subdomains coincide, that is, $\mathbf{B}_f\hat{\mathbf{p}} = 0 \iff \hat{\mathbf{p}} = \mathbf{N}_f\mathbf{p}$ for some $\mathbf{p}$, and $\mathbf{B}_e\hat{\mathbf{u}} = 0 \iff \hat{\mathbf{u}} = \mathbf{N}_e\mathbf{u}$ for some $\mathbf{u}$. See [8] for details on the construction of such matrices with entries $0, \pm 1$. Here, we use the matrices from [8] and orthogonalize their rows for numerical stability; the resulting matrices are still sparse.

Multiplying the second equation in (2.6) by $\omega^2\rho_f$ to symmetrize the off-diagonal block and introducing Lagrange multipliers λ_f and λ_e for the constraints $\mathbf{B}_f\mathbf{p} = 0$ and $\mathbf{B}_e\mathbf{u} = 0$, we get the system of linear equations in block form,

$$\begin{bmatrix} -\hat{\mathbf{K}}_f + k^2\hat{\mathbf{M}}_f - ik\hat{\mathbf{G}} & -\omega^2\rho_f\hat{\mathbf{T}} & \mathbf{B}_f' & 0 \\ -\omega^2\rho_f\hat{\mathbf{T}}' & \omega^2\rho_f(-\hat{\mathbf{K}}_e + \omega^2\hat{\mathbf{M}}_e) & 0 & \mathbf{B}_e' \\ \mathbf{B}_f & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (3.4)$$

where $\mathbf{N}'\hat{\mathbf{r}} = \mathbf{r}$. Similarly as in [8], it can be shown that the system (3.4) is equivalent to (2.6) in the sense that $(\mathbf{p}, \mathbf{u})$ is a solution of (2.6) if and only if $(\hat{\mathbf{p}}, \hat{\mathbf{u}}, \lambda_f, \lambda_e)$ with $\hat{\mathbf{p}} = \mathbf{N}_f\mathbf{p}$, $\hat{\mathbf{u}} = \mathbf{N}_e\mathbf{u}$, is a solution of (3.4) for some λ_f and λ_e.

Using the properties of the global to local maps $\mathbf{N}_f$ and $\mathbf{N}_e$, it is easy to see that $(\hat{\mathbf{p}}, \hat{\mathbf{u}}, \lambda_f, \lambda_e)$ is a solution of (3.4) if and only if $\hat{\mathbf{p}} = \mathbf{N}_f\mathbf{p}$ and $\hat{\mathbf{u}} = \mathbf{N}_e\mathbf{u}$, where $(\mathbf{p}, \mathbf{u})$ solves (2.6).

We will want to eventually eliminate the variables $\hat{\mathbf{p}}$ and $\hat{\mathbf{u}}$. But the matrices $-\hat{\mathbf{K}}_f + k^2\hat{\mathbf{M}}_f$ and $-\hat{\mathbf{K}}_e + \omega^2\hat{\mathbf{M}}_e$ may be singular due to resonance. For this reason, the continuity of normal derivative and traction between subdomains are replaced by artificial radiation conditions,

$$p^s + i\sigma^{st}k\frac{\partial p^s}{\partial \nu^s} = u^t + i\sigma^{ts}k\frac{\partial p^t}{\partial \nu^t} \text{ on } \partial\Omega_f^s \cap \partial\Omega_f^t \tag{3.5}$$

and

$$u^s + i\sigma^{ts}\alpha\tau(u^s)\nu^s = u^t + i\sigma^{st}\alpha\tau(u^t)\nu^t, \text{ on } \partial\Omega_e^s \cap \partial\Omega_e^t. \tag{3.6}$$

Here, $\sigma^{st} = \pm 1$ or 0, $\sigma^{st} = -\sigma^{ts}$, and

$$\alpha = \alpha_0\omega\sqrt{\rho_e(\lambda + 2\mu)}. \tag{3.7}$$

If $\sigma^{st} = \pm 1$, the interface condition (3.5) allows a plane wave to pass in one normal direction through the interface between the subdomains. Similarly, by a simple computation, the condition (3.6) with $\alpha_0 = 1$ is satisfied by the plane pressure wave

$$u(x) = de^{i\frac{\omega}{c_p}d\cdot x}, \quad |d| = 1, \quad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho_e}}, \tag{3.8}$$

in one of the normal directions, $d = \pm\nu^s$. An alternative form of (3.6) is

$$\nu^s(u^s\cdot\nu^s) + i\sigma^{ts}\alpha\tau(u^s)\nu^s = \nu^t(u^t\cdot\nu^t) + i\sigma^{st}\alpha\tau(u^t)\nu^t \text{ on } \partial\Omega_f^s \cap \partial\Omega_f^t, \tag{3.9}$$

which, for α from (3.7) with $\alpha_0 = 1$, is also satisfied by the pressure wave (3.8) in normal direction. In [9], the condition (3.9) with $\alpha = \omega\rho_e$ was used.

This change of intersubdomain interface conditions corresponds to replacing the subdomain matrices $-\hat{\mathbf{K}}_f + k^2\hat{\mathbf{M}}_f$ and $-\hat{\mathbf{K}}_e + \omega^2\hat{\mathbf{M}}_e$ by *regularized matrices*

$$\begin{aligned}\hat{\mathbf{A}}_f &= -\hat{\mathbf{K}}_f + k^2\hat{\mathbf{M}}_f + ik\hat{\mathbf{G}}_f + \hat{\mathbf{R}}_f,\\ \hat{\mathbf{A}}_e &= -\hat{\mathbf{K}}_e + \omega^2\hat{\mathbf{M}}_e + \hat{\mathbf{R}}_e,\end{aligned}$$

where the regularization matrices are given by

$$\hat{\mathbf{R}}_f = \text{diag}(\mathbf{R}_f^s), \qquad \mathbf{p}^{\mathbf{s}'}\mathbf{R}_f^s\mathbf{q}^s = ik\sum_{\substack{t=1\\ t\neq s}}^{N_f}\sigma^{st}\int_{\partial\Omega_f^s\cap\partial\Omega_f^t} pq$$

between fluid subdomains, and

$$\hat{\mathbf{R}}_e = \text{diag}(\mathbf{R}_e^s), \qquad \mathbf{u}^{\mathbf{s}'}\mathbf{R}_e^s\mathbf{v}^s = i\alpha\sum_{\substack{t=1\\ t\neq s}}^{N_e}\sigma^{st}\int_{\partial\Omega_e^s\cap\partial\Omega_e^t} u\cdot v, \tag{3.10}$$

between elastic subdomains for the interface condition (3.6) and by

$$\hat{\mathbf{R}}_e = \mathrm{diag}(\mathbf{R}_e^s), \qquad \mathbf{u}^{\mathbf{s}\prime}\mathbf{R}_e^s\mathbf{v}^s = i\alpha \sum_{\substack{t=1 \\ t\neq s}}^{N_e} \sigma^{st} \int\limits_{\partial\Omega_e^s \cap \partial\Omega_e^t} (\nu^s \cdot u)(\nu^s \cdot v), \tag{3.11}$$

if (3.9) is used.

It is shown in [6] for the Helmholtz equation that if for a given s, all $\sigma^{st} \geq 0$ or all $\sigma^{st} \leq 0$ with some $\sigma^{st} \neq 0$, then $\hat{\mathbf{A}}_f^s$ is invertible. The case of elastic subdomains is similar. For details on strategies for choosing σ^{st} to guarantee this, see [6]. In our computations, we simply choose $\sigma^{st} = +1$ if $s > t$, $\sigma_f^{st} = -1$ if $s < t$.

Because

$$\mathbf{N}_f'\hat{\mathbf{R}}_f\mathbf{N}_f = 0, \qquad \mathbf{N}_e'\hat{\mathbf{R}}_e\mathbf{N}_e = 0,$$

the effect of adding the matrices $\hat{\mathbf{R}}_f$, $\hat{\mathbf{R}}_f$ cancels in the assembled system, and the system (3.4) is equivalent to

$$\begin{bmatrix} \hat{\mathbf{A}}_f & -\omega^2\rho_f\hat{\mathbf{T}} & \mathbf{B}_f' & 0 \\ -\omega^2\rho_f\hat{\mathbf{T}}' & \omega^2\rho_f\hat{\mathbf{A}}_e & 0 & \mathbf{B}_e' \\ \mathbf{B}_f & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.12}$$

Eliminating the original degrees of freedom at this point does not result in independent computation in each subdomain, because of coupling of degrees of freedom across the wet interface by the matrix $\hat{\mathbf{T}}$. Hence, we first duplicate the interface degrees of freedom as follows. Since the value of $\hat{\mathbf{T}}\hat{\mathbf{u}}$ depends on the values of $\hat{\mathbf{u}}$ on Γ only, we have

$$\hat{\mathbf{T}}\hat{\mathbf{u}} = \hat{\mathbf{T}}\mathbf{J}_e\hat{\mathbf{u}}_\Gamma, \qquad \hat{\mathbf{u}}_\Gamma = \mathbf{J}_e'\hat{\mathbf{u}},$$

where $\hat{\mathbf{J}}_e$ is the matrix of the operator of embedding a subvector that corresponds to degrees of freedom on Γ into $\hat{\mathbf{u}}$ by adding zero entries. Similarly,

$$\hat{\mathbf{T}}'\hat{\mathbf{p}} = \hat{\mathbf{T}}'\mathbf{J}_f\hat{\mathbf{p}}_\Gamma, \qquad \hat{\mathbf{p}}_\Gamma = \mathbf{J}_f'\hat{\mathbf{p}}.$$

Therefore, we obtain the augmented system equivalent to (3.12),

$$\begin{bmatrix} \hat{\mathbf{A}}_f & 0 & \mathbf{B}_f' & 0 & 0 & -\omega^2\rho_f\hat{\mathbf{T}}\mathbf{J}_e \\ 0 & \omega^2\rho_f\hat{\mathbf{A}}_e & 0 & \mathbf{B}_e' & -\omega^2\rho_f\hat{\mathbf{T}}'\mathbf{J}_f & 0 \\ \mathbf{B}_f & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{B}_e & 0 & 0 & 0 & 0 \\ \mathbf{J}_f' & 0 & 0 & 0 & -\mathbf{I} & 0 \\ 0 & \mathbf{J}_e' & 0 & 0 & 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{u}} \\ \lambda_f \\ \lambda_e \\ \hat{\mathbf{p}}_\Gamma \\ \hat{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{r}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.13}$$

Because the variables in a coupled system typically have vastly different scales, we use symmetric diagonal scaling to get the scaled system

$$\begin{bmatrix} \tilde{\mathbf{A}}_f & 0 & \tilde{\mathbf{B}}_f' & 0 & 0 & -\tilde{\mathbf{T}}\mathbf{J}_e \\ 0 & \tilde{\mathbf{A}}_e & 0 & \tilde{\mathbf{B}}_e' & -\tilde{\mathbf{T}}'\mathbf{J}_f & 0 \\ \tilde{\mathbf{B}}_f & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\mathbf{B}}_e & 0 & 0 & 0 & 0 \\ \mathbf{J}_f' & 0 & 0 & 0 & -\mathbf{I} & 0 \\ 0 & \mathbf{J}_e' & 0 & 0 & 0 & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{p}} \\ \tilde{\mathbf{u}} \\ \tilde{\lambda}_f \\ \tilde{\lambda}_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{r}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{3.14}$$

where the matrices and the vectors scale as $\tilde{\mathbf{A}}_f = \mathbf{D}_f\hat{\mathbf{A}}_f\mathbf{D}_f$, $\tilde{\mathbf{A}}_e = \omega^2\rho_f\mathbf{D}_e\hat{\mathbf{A}}_e\mathbf{D}_e$, $\tilde{\mathbf{T}} = \omega^2\rho_f\mathbf{D}_f\hat{\mathbf{T}}\mathbf{D}_e$, $\tilde{\mathbf{B}}_f = \mathbf{E}_f\mathbf{B}_f\mathbf{D}_f$, $\tilde{\mathbf{B}}_e = \mathbf{E}_e\mathbf{B}_e\mathbf{D}_e$, $\tilde{\mathbf{r}} = \mathbf{D}_f\hat{\mathbf{r}}$, $\tilde{\mathbf{p}} = \mathbf{D}_f\hat{\mathbf{p}}$, $\tilde{\mathbf{u}} = \mathbf{D}_e\hat{\mathbf{u}}$, $\lambda_f = \mathbf{D}_f\tilde{\lambda}_f$, $\lambda_e = \mathbf{D}_e\tilde{\lambda}_e$. The scaling matrices $\mathbf{D}_f$, $\mathbf{D}_e$, $\mathbf{E}_f$, and $\mathbf{E}_e$, are diagonal. We have chosen scaling matrices with positive diagonal entries such that the absolute values of the diagonal entries of $\tilde{\mathbf{A}}_f$ and $\tilde{\mathbf{A}}_e$ are one and the ℓ^2 norms of the columns of $\tilde{\mathbf{B}}_e$ and $\tilde{\mathbf{B}}_f$ are one.

Computing $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{u}}$ from the first two equations in (3.14) gives

$$\tilde{\mathbf{p}} = \tilde{\mathbf{A}}_f^{-1}(\tilde{\mathbf{r}} - \tilde{\mathbf{B}}_f'\tilde{\lambda}_f + \tilde{\mathbf{T}}\mathbf{J}_e\tilde{\mathbf{u}}_\Gamma) \tag{3.15}$$

$$\tilde{\mathbf{u}} = \tilde{\mathbf{A}}_e^{-1}(-\tilde{\mathbf{B}}_e'\tilde{\lambda}_e + \tilde{\mathbf{T}}'\mathbf{J}_f\tilde{\mathbf{p}}_\Gamma) \tag{3.16}$$

Substituting $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{u}}$ from (3.15), (3.16) into the rest of the equations in (3.14), we obtain the reduced system

$$\mathbf{F}\mathbf{x} = \mathbf{b}, \tag{3.17}$$

where

$$\mathbf{F} = \begin{bmatrix} \tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f' & 0 & 0 & -\tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{T}}\mathbf{J}_e \\ 0 & \tilde{\mathbf{B}}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{B}}_e & -\tilde{\mathbf{B}}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{T}}'\mathbf{J}_f & \\ -\mathbf{J}_f'\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f' & 0 & -\mathbf{I} & \mathbf{J}_f'\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{T}}\mathbf{J}_e \\ 0 & -\mathbf{J}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{B}}_e & \mathbf{J}_e\tilde{\mathbf{A}}_e^{-1}\tilde{\mathbf{T}}'\mathbf{J}_f & -I \end{bmatrix}, \tag{3.18}$$

and

$$\mathbf{x} = \begin{bmatrix} \lambda_f \\ \lambda_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{r}} \\ 0 \\ -\mathbf{J}_f'\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{r}} \\ 0 \end{bmatrix}.$$

In equation (3.18), the first diagonal block $\tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f'$ is exactly same as in the FETI-H method for the Helmholtz equation. The second diagonal block $\tilde{\mathbf{B}}_f\tilde{\mathbf{A}}_f^{-1}\tilde{\mathbf{B}}_f'$ is the analogue of FETI-H for the elastodynamic problem.

Evaluating the matrix vector product $\mathbf{F}\mathbf{x}$ requires the solution of one independent problem per subdomain, because

$$\mathbf{F}\begin{bmatrix} \lambda_f \\ \lambda_e \\ \tilde{\mathbf{p}}_\Gamma \\ \tilde{\mathbf{u}}_\Gamma \end{bmatrix} = \begin{bmatrix} -\tilde{\mathbf{B}}_f\tilde{\mathbf{q}} \\ -\tilde{\mathbf{B}}_e\tilde{\mathbf{v}} \\ \mathbf{J}_f'\tilde{\mathbf{q}} - \tilde{\mathbf{p}}_\Gamma \\ \mathbf{J}_e'\tilde{\mathbf{v}} - \tilde{\mathbf{u}}_\Gamma \end{bmatrix}, \quad \text{where} \quad \begin{cases} \tilde{\mathbf{q}} = \tilde{\mathbf{A}}_f^{-1}(-\tilde{\mathbf{B}}_f'\tilde{\lambda}_f + \tilde{\mathbf{T}}\mathbf{J}_e\tilde{\mathbf{u}}_\Gamma), \\ \tilde{\mathbf{v}} = \tilde{\mathbf{A}}_e^{-1}(-\tilde{\mathbf{B}}_e'\tilde{\lambda}_e + \tilde{\mathbf{T}}'\mathbf{J}_f\tilde{\mathbf{p}}_\Gamma). \end{cases}$$

The iterative method then consists of solving the linear system (3.17) by GMRES preconditioned by a subspace correction as follows. Let $\mathbf{Q}$ be a matrix with the same number of rows as $\mathbf{F}$ and linearly independent columns. The columns of $\mathbf{Q}$ form the basis of the *coarse space*. The orthogonality condition

$$\mathbf{Q}'(\mathbf{F}\mathbf{x} - \mathbf{b}) = 0, \tag{3.19}$$

is enforced through the iterations by adding a correction from the coarse space in each iteration. That is, GMRES is applied to the preconditioned system

$$\mathbf{P}\mathbf{F}\mathbf{x} = \mathbf{P}\mathbf{b}, \tag{3.20}$$

where $\mathbf{P} = (\mathbf{I} - \mathbf{Q}(\mathbf{Q}'\mathbf{F}\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{F})$ and the initial approximation $\mathbf{x} = \mathbf{Q}(\mathbf{Q}'\mathbf{F}\mathbf{Q})^{-1}\mathbf{b}$ satisfies (3.19). Because the increments are in the range of $\mathbf{P}$ and $\mathbf{Q}'\mathbf{F}\mathbf{P} = 0$, all iterates satisfy (3.19).

We choose the matrix $\mathbf{Q}$ of the form

$$\begin{bmatrix} \mathbf{D}_f\mathbf{B}_f\mathrm{diag}(\mathbf{Y}_f^s)_s & 0 & 0 & 0 \\ 0 & \mathbf{D}_e\mathbf{B}_e\mathrm{diag}(\mathbf{Y}_e^s)_s & 0 & 0 \\ 0 & 0 & \mathbf{D}_f\mathbf{J}_f'\mathrm{diag}(\mathbf{Z}_f^s)_s & 0 \\ 0 & 0 & 0 & \mathbf{D}_e\mathbf{J}_e'\mathrm{diag}(\mathbf{Z}_e^s)_s \end{bmatrix}. \quad (3.21)$$

and orthogonalize its columns by the QR algorithm. For a fluid subdomain Ω_f^s, we choose $\mathbf{Y}_f^s$ as the matrix of columns that are discrete representations of plane waves in a small number of equally distributed directions, or discrete representation of the constant function. For a solid subdomain Ω_e^s, the columns of $\mathbf{Y}_e^s$ are discrete representations of plane pressure and shear waves, or of rigid body motions. The matrices $\mathbf{Z}_f^s$ and $\mathbf{Z}_e^s$ are chosen in the same way as $\mathbf{Y}_f^s$ and $\mathbf{Y}_e^s$. See [9] for further details and a discussion why the method for the coupled problem can be expected to perform about as FETI-H for the fluid and the elastic parts separately.

4. Radiation-Like Condition on the Wet Interface. For some frequencies, the matrix $-\mathbf{K}_e + \omega^2\mathbf{M}_e$ in the coupled system (2.6) will be singular. The inverse of this matrix is required by the method if there is only one elastic subdomain. Therefore, in this case, we replace (2.6) by an equivalent system, obtained by adding to the second block of equations a linear combination of the first block in such a way that the term added to $-\mathbf{K}_e + \omega^2\mathbf{M}_e$ resembles a radiation condition:

$$\begin{bmatrix} -\mathbf{K}_f + k^2\mathbf{M}_f - ik\mathbf{G}_f & -\rho_f\omega^2\mathbf{T} \\ -\mathbf{T}' + i\frac{\beta}{-\rho_f\omega^2}\mathbf{T}'(-\mathbf{K}_f + k^2\mathbf{M}_f - ik\mathbf{G}_f) & -\mathbf{K}_e + \omega^2\mathbf{M}_e + i\beta\mathbf{T}'\mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{u} \end{bmatrix} = \mathbf{y}, \quad (4.1)$$

where

$$\mathbf{y} = \begin{bmatrix} \mathbf{r} \\ i\frac{\beta}{-\rho_f\omega^2}\mathbf{T}'r \end{bmatrix}.$$

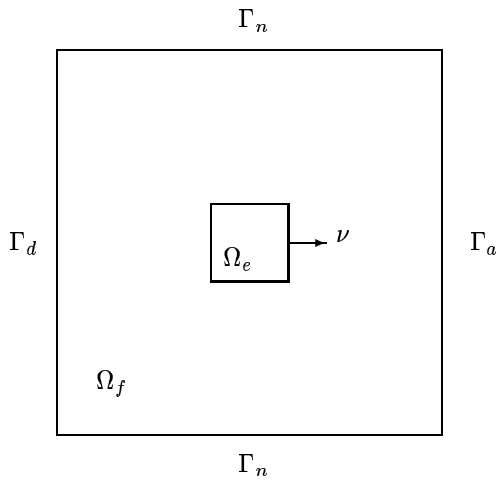
To obtain an added term with consistent physical units and similar to the artificial radiation condition (3.10), we choose

$$\beta = \beta_0\frac{\omega\sqrt{\rho_e(\lambda + 2\mu)}}{\|\mathbf{T}\|_1}. \quad (4.2)$$

In the case of more than several fluid subdomains and one elastic subdomain, this process is easily implemented using the local subdomain matrices for $\mathbf{T}$. For more than one elastic subdomain, computational experiments indicate that introducing an artificial radiation condition on the wet interface is not necessary.

5. Computational results. Computational results showing scalability of the method were presented in [9]. Here, we focus on the performance of the method when k equals or is close to a value that makes some of the subdomain matrices singular, and for different choices of the radiation-like condition between elastic subdomains.

Figure 5.1: Model 2D Problem



We consider a model 2D problem with a scatterer in the center a waveguide, cf., Fig. 5.1. The fluid domain Ω_f is a square with side $1\,m$, filled with water with density $\rho_f = 1000\,kg\,m^{-3}$ and speed of sound $c_f = 1500\,m\,s^{-1}$. The scatterer is a square in the center of the fluid domain, consisting of aluminum with density $\rho_e = 2700\,kg\,m^{-3}$ and Lamé elasticity coefficients $\lambda = 5.5263.10^{10}\,N\,m^{-2}$, $\mu = 2.595.10^{10}\,N\,m^{-2}$. The domain is discretized with a mesh of 200 by 200 bilinear elements. The coarse space consists of 8 plane waves in the fluid subdomains and 4 plane pressure waves and 4 plane shear waves in the solid subdomains (the first two blocks of the matrix $\mathbf{Q}$ in (3.21)). The same number of coarse space functions is used for the coarse space for the wet interface (the last two blocks of the matrix $\mathbf{Q}$). The iterations are terminated when the relative residual reached 10^{-6}. Then the scaled residual in the original variables are checked,

$$Res = \max_i \frac{|\mathbf{d}_i - \sum_j \mathbf{K}_{ij}\mathbf{z}_j|}{\sum_j |\mathbf{K}_{ij}|\,|\mathbf{z}_j|}, \tag{5.1}$$

where $\mathbf{z}$, $\mathbf{K}$, and $\mathbf{d}$ are the solution vector, the matrix, and the right-hand-side, respectively, of the coupled system (2.6). In all cases when the iterations converged, this scaled residual was of the order 10^{-6} to 10^{-7}.

Figure 5.2 shows the number of iterations for varying constant α_0 in the artificial radiation condition (3.6) between elastic subdomains. The scatter was size 0.4 by 0.4 and the fluid and the elastic domains were decomposed into 4 subdomains each. One can see that the number of iterations for $\alpha_0 = 1$ is slightly larger over all frequencies, while the iterations diverge for frequencies equal to or very close the resonance frequencies for $\alpha_0 \leq 10^{-4}$.

The number of iterations for the same test problem and the artifical radiation condition (3.9) was almost exactly same (not shown).

Figure 5.3 reports the number of iterations for the same problem with the artificial

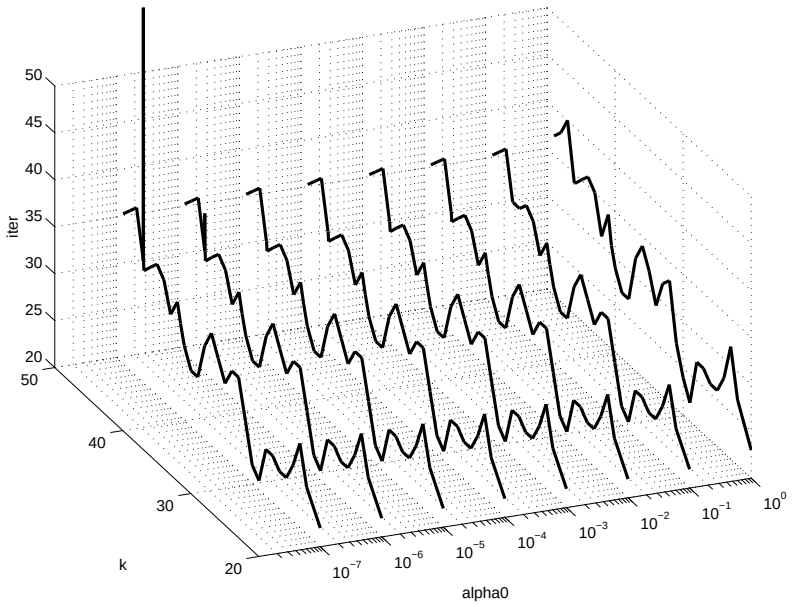


Figure 5.2: Number of iterations for different α_0 in artificial radiation condition (3.6) between elastic subdomains

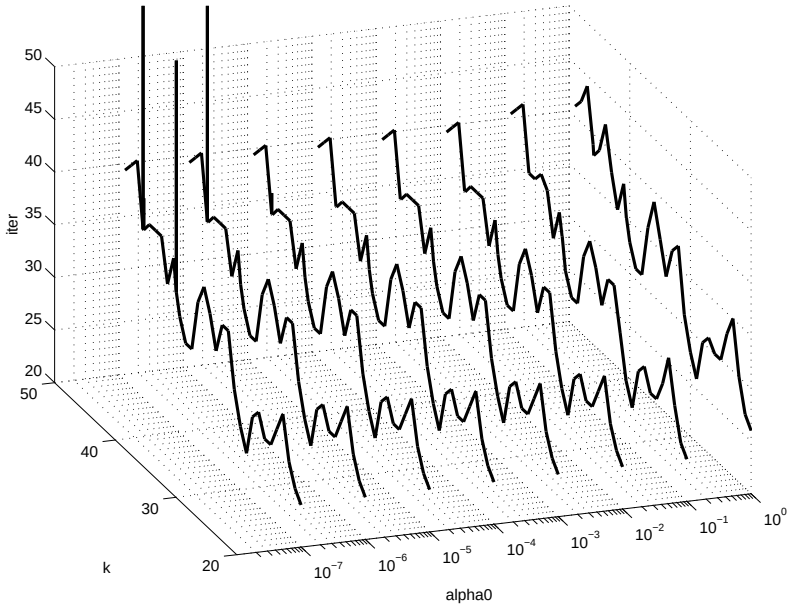


Figure 5.3: Number of iterations for different α_0 in artificial radiation condition (3.6) between elastic subdomains and selection of basis instead of orthogonalization of the rows of $\mathbf{B}$ and the columns of $\mathbf{Q}$

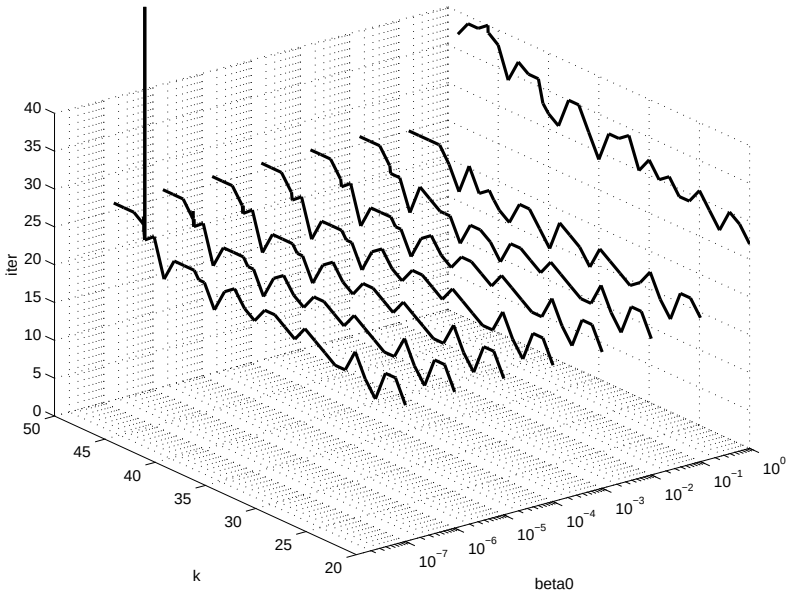


Figure 5.4: Number of iterations for different β_0 in artificial radiation on wet interface by (4.1) and (4.2)

radiation condition (3.6) between elastic subdomains, but instead of orthogonalization of the rows of the matrices $\mathbf{B}$ and the columns of the matrix $\mathbf{Q}$, bases are selected as linearly independent subsets. There are more iterations required and divergence occurs for more frequencies and larger values of the parameter α_0.

Figure 5.4 reports the number of iterations for decreasing strength β_0 of the artificial radiation-like term on the wet interface. The scatterer was size 0.2 by 0.2, forming one elastic subdomain, and the fluid domain was decomposed along the midlines of the square into 4 subdomains. One can see that the choice $\beta_0 = 1$ increases the number of iterations significantly over all frequencies, while for $\beta_0 = 10^{-5}$, the iterations diverge for frequencies equal to or close to a resonance frequency. The elastic subdomain in this experiment is of the same size as the elastic subdomain for the examples in Figure 5.2, so the resonance frequencies are same.

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